

THIRD HOMOLOGY OF SL_2 AND THE INDECOMPOSABLE K_3

BEHROOZ MIRZAI

ABSTRACT. It is known that, for an infinite field F , the indecomposable part of $K_3(F)$ and the third homology of $SL_2(F)$ are closely related. In fact, there is a canonical map $\alpha : H_3(SL_2(F), \mathbb{Z})_{F^*} \rightarrow K_3(F)^{\text{ind}}$. Suslin has raised the question that, is α an isomorphism [9]? Recently Hutchinson and Tao have shown that this map is surjective [4]. They also gave some arguments about its injectivity. In this article, we improve their arguments and show that α is bijective if and only if the natural maps $H_3(GL_2(F), \mathbb{Z}) \rightarrow H_3(GL_3(F), \mathbb{Z})$ and $H_3(SL_2(F), \mathbb{Z})_{F^*} \rightarrow H_3(GL_2(F), \mathbb{Z})$ are injective.

INTRODUCTION

For an infinite field F , Suslin has proved that the Hurewicz homomorphism

$$h_3 : K_3(F) = \pi_3(BSL(F)^+) \longrightarrow H_3(BSL(F)^+, \mathbb{Z}) \simeq H_3(SL(F), \mathbb{Z})$$

is surjective with 2-torsion kernel. In fact, he has shown that h_3 sits in the exact sequence

$$K_2(F) \xrightarrow{l(-1)} K_3(F) \longrightarrow H_3(SL(F), \mathbb{Z}) \longrightarrow 0,$$

where the homomorphism $l(-1) : K_2(F) \rightarrow K_3(F)$ coincides with multiplication by $l(-1) \in K_1(\mathbb{Z})$ [10, Lemma 5.2, Corollary 5.2]. Let

$$\alpha : H_0(F^*, H_3(SL_2(F), \mathbb{Z})) \rightarrow K_3(F)^{\text{ind}}$$

be the composition of the following sequence of homomorphisms

$$\begin{aligned} H_0(F^*, H_3(SL_2(F), \mathbb{Z})) &\xrightarrow{\text{inc}_*} H_3(SL(F), \mathbb{Z}) \xrightarrow{\bar{h}_3^{-1}} K_3(F)/l(-1)K_2(F) \\ &\xrightarrow{p} K_3(F)^{\text{ind}} := K_3(F)/K_3^M(F), \end{aligned}$$

where inc_* is induced by the inclusion $\text{inc} : SL_2(F) \rightarrow SL(F)$, and p is induced by the inclusion $l(-1)K_2(F) \subseteq \text{im}(K_3^M(F) \rightarrow K_3(F))$. For algebraically closed fields, it was known that α is an isomorphism [1], [9]. Following this, Suslin raised the following question:

Question (Suslin). Is it true that $H_0(F^*, H_3(SL_2(F), \mathbb{Z}))$ coincides with $K_3(F)^{\text{ind}}$? (See [9, Question 4.4]).

In other words, is α bijective for an arbitrary infinite field F ? This question is true after killing 2-power torsion elements (i.e. after tensoring the both side of this map with $\mathbb{Z}[1/2]$) or when $F^* = F^{*2} = \{a^2 | a \in F^*\}$ [6, Proposition 6.4].

Recently Hutchinson and Tao have proved that α is surjective [4, Lemma 5.1]. At the end of their article, they also stated that the injectivity of α is equivalent to the following three conditions:

- 1) $\text{im}(H_3(\text{SL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{SL}(F), \mathbb{Z})) \simeq K_3(F)^{\text{ind}}$,
- 2) $H_3(\text{GL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{GL}_3(F), \mathbb{Z})$ is injective,
- 3) $H_0(F^*, H_3(\text{SL}_2(F), \mathbb{Z})) \rightarrow H_3(\text{GL}_2(F), \mathbb{Z})$ is injective.

Among these conditions, the first one seems very unnatural. It is also very strong condition. In this article we show that this condition can be dropped from this list. In fact we prove the following result.

Theorem. *Let F be an infinite field. The following conditions are equivalent.*

- (i) *The homomorphism $\alpha : H_0(F^*, H_3(\text{SL}_2(F), \mathbb{Z})) \rightarrow K_3(F)^{\text{ind}}$ is bijective.*
- (ii) *The natural homomorphisms $H_3(\text{GL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{GL}_3(F), \mathbb{Z})$ and $H_0(F^*, H_3(\text{SL}_2(F), \mathbb{Z})) \rightarrow H_3(\text{GL}_2(F), \mathbb{Z})$ are injective.*

We should mention that in the mean time we also establish that the kernel of the homomorphism

$$H_3(\text{inc}) : H_3(\text{GL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{GL}_3(F), \mathbb{Z})$$

is equal to

$$\text{im}(H_3(\text{SL}_2(F), \mathbb{Z}) \rightarrow H_3(\text{GL}_2(F), \mathbb{Z})) \cap F^* \cup H_3(\text{GL}_1(F), \mathbb{Z}),$$

where the cup product is induced by the natural diagonal inclusion $\text{inc} : F^* \times \text{GL}_1(F) \rightarrow \text{GL}_2(F)$. In another article we have shown that the kernel of $H_3(\text{inc})$ is a 2-torsion group [7] (see Theorem 2.1 below).

It seems that, for an arbitrary field, not much is known about the kernel of

$$H_0(F^*, H_3(\text{SL}_2(F), \mathbb{Z})) \rightarrow H_3(\text{GL}_2(F), \mathbb{Z}),$$

except that it is a 2-power torsion group (see proof of Theorem 6.1 in [6]).

Notation. In this article by $H_i(G)$ we mean the homology of group G with integral coefficients, namely $H_i(G, \mathbb{Z})$. By GL_n (resp. SL_n) we mean the general (resp. special) linear group $\text{GL}_n(F)$ (resp. $\text{SL}_n(F)$), where F is an infinite field. If $A \rightarrow A'$ is a homomorphism of abelian groups, by A'/A we mean $\text{coker}(A \rightarrow A')$ and we take other liberties of this kind. Here by Σ_n we mean the symmetric group of rank n .

1. THE GROUP $H_1(F^*, H_2(\mathrm{SL}_2))$

We start this section by looking at the corresponding Lyndon/Hochschild-Serre spectral sequence of the commutative diagram of extensions

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{SL}_2 & \longrightarrow & \mathrm{GL}_2 & \xrightarrow{\det} & F^* \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathrm{SL}_3 & \longrightarrow & \mathrm{GL}_3 & \xrightarrow{\det} & F^* \longrightarrow 1. \end{array}$$

So we get a morphism of spectral sequences

$$\begin{array}{ccc} \mathcal{E}_{p,q}^2 = H_p(F^*, H_q(\mathrm{SL}_2)) & \Longrightarrow & H_{p+q}(\mathrm{GL}_2), \\ \downarrow & & \downarrow \\ E_{p,q}^2 = H_p(F^*, H_q(\mathrm{SL}_3)) & \Longrightarrow & H_{p+q}(\mathrm{GL}_3). \end{array}$$

By an easy analysis of this spectral sequence we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} H_3(\mathrm{SL}_2)_{F^*} & \longrightarrow & H_3(\mathrm{GL}_2)/H_3(\mathrm{GL}_1) & \xrightarrow{\varphi} & H_1(F^*, H_2(\mathrm{SL}_2)) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_3(\mathrm{SL}_3)_{F^*} & \longrightarrow & H_3(\mathrm{GL}_3)/H_3(\mathrm{GL}_1) & \xrightarrow{\psi} & H_1(F^*, H_2(\mathrm{SL}_3)) \longrightarrow 0. \end{array}$$

The following theorem is due to Hutchinson and Tao [4, Theorem 3.2], which is very fundamental in their proof of the surjectivity of α .

Theorem 1.1. *The inclusion $\mathrm{SL}_2 \longrightarrow \mathrm{SL}_3$ induces a short exact sequence*

$$0 \longrightarrow H_1(F^*, H_2(\mathrm{SL}_2)) \longrightarrow H_1(F^*, H_2(\mathrm{SL}_3)) \longrightarrow k_3^M(F) \longrightarrow 0,$$

where $k_3^M(F) := K_3^M(F)/2$.

Since the action of F^* on $H_2(\mathrm{SL}_3)$ is trivial,

$$H_1(F^*, H_2(\mathrm{SL}_3)) \simeq F^* \otimes K_2^M(F).$$

So we consider $H_1(F^*, H_2(\mathrm{SL}_2))$ as a subgroup of $F^* \otimes K_2^M(F)$. It is easy to see that the map

$$H_1(F^*, H_2(\mathrm{SL}_3)) \longrightarrow k_3^M(F)$$

is induced by the natural product map $F^* \otimes K_2^M(F) \longrightarrow K_3^M(F)$. Since the n -th Milnor K -group, $K_n^M(F)$, is naturally isomorphic to the n -th tensor of F^* modulo the two families of relations

$$a_1 \otimes \cdots \otimes a_{n-1} \otimes (1 - a_{n-1}), \quad a_i \in F^*, a_{n-1} \neq 1,$$

$$a_1 \otimes \cdots \otimes a_i \otimes a_{i+1} \cdots \otimes a_n + a_1 \otimes \cdots \otimes a_{i+1} \otimes a_i \cdots \otimes a_n, \quad a_i \in F^*,$$

it easily follows that the kernel of the product map $F^* \otimes K_2^M(F) \longrightarrow K_3^M(F)$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$. This proves the following lemma.

Lemma 1.2. *As a subgroup of $H_1(F^*, H_2(\mathrm{SL}_3)) = F^* \otimes K_2^M(F)$, the group $H_1(F^*, H_2(\mathrm{SL}_2))$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$ and $2d \otimes \{e, f\}$.*

To go further, we need to introduce some notations. Let G be a group and set

$$\mathbf{c}(g_1, g_2, \dots, g_n) := \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) [g_{\sigma(1)} | g_{\sigma(2)} | \dots | g_{\sigma(n)}] \in H_n(G),$$

where $g_i \in G$ pairwise commute and S_n is the symmetric group of degree n . Here we use the bar resolution of G [2, Chapter I, Section 5] to define the homology of G .

Lemma 1.3. *Let G and G' be two groups.*

(i) *If $h_1 \in G$ commutes with all the elements $g_1, \dots, g_n \in G$, then*

$$\mathbf{c}(g_1 h_1, g_2, \dots, g_n) = \mathbf{c}(g_1, g_2, \dots, g_n) + \mathbf{c}(h_1, g_2, \dots, g_n).$$

(ii) *For every $\sigma \in S_n$, $\mathbf{c}(g_{\sigma(1)}, \dots, g_{\sigma(n)}) = \mathrm{sign}(\sigma) \mathbf{c}(g_1, \dots, g_n)$.*

(iii) *The cup product of $\mathbf{c}(g_1, \dots, g_p) \in H_p(G)$ and $\mathbf{c}(g'_1, \dots, g'_q) \in H_q(G')$ is $\mathbf{c}((g_1, 1), \dots, (g_p, 1), (1, g'_1), \dots, (1, g'_q)) \in H_{p+q}(G \times G')$.*

Proof. The proof follows from direct computations, so we leave it to the interested readers. \square

2. THE KERNEL OF $H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$

For simplicity, in the rest of this article, we use the following notation

$$k_{a,b,c} := \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)) \in H_3(\mathrm{GL}_2).$$

The following theorem has been proved in [7, Theorem 3.1].

Theorem 2.1. *The kernel of $\mathrm{inc}_{1*} : H_3(\mathrm{GL}_2) \longrightarrow H_3(\mathrm{GL}_3)$ consists of elements of the form $\sum k_{a,b,c} + k_{b,a,c}$ such that*

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in F^* \otimes K_2^M(F).$$

In particular $\ker(\mathrm{inc}_{1}) \subseteq F^* \cup H_2(\mathrm{GL}_1) \subseteq H_3(\mathrm{GL}_2)$, where the cup product is induced by the natural diagonal inclusion $F^* \times \mathrm{GL}_1 \longrightarrow \mathrm{GL}_2$. Moreover $\ker(\mathrm{inc}_{1*})$ is a 2-torsion group.*

Let Ψ and Φ be the following compositions,

$$F^* \otimes K_2^M(F) \xrightarrow{\mathrm{id}_{F^*} \otimes \iota} F^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3(F^* \times \mathrm{GL}_2) \xrightarrow{\mathrm{inc}_*} H_3(\mathrm{GL}_3),$$

$$F^* \otimes K_2^M(F) \xrightarrow{\mathrm{id}_{F^*} \otimes \iota} F^* \otimes H_2(\mathrm{GL}_2) \xrightarrow{\cup} H_3(F^* \times \mathrm{GL}_2) \xrightarrow{\beta_*} H_3(\mathrm{GL}_2),$$

respectively, where $\iota : K_2^M(F) \simeq H_2(\mathrm{SL}_2)_{F^*} \longrightarrow H_2(\mathrm{GL}_2)$ is the natural inclusion given by the formula $\{a, b\} \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(b, b^{-1}))$ [3, Proposition A.11] and $\beta : F^* \times \mathrm{GL}_2 \longrightarrow \mathrm{GL}_2$ is given by $(a, A) \mapsto aA$. It is easy to see that

$$\Psi(a \otimes \{b, c\}) = \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, b, 1), \mathrm{diag}(1, c, c^{-1})),$$

$$\Phi(a \otimes \{b, c\}) = \mathbf{c}(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}(c, c^{-1})).$$

Lemma 2.2. *Let Θ be the composition*

$$H_3(GL_2)/H_3(GL_1) \xrightarrow{\varphi} H_1(F^*, H_2(SL_2)) \hookrightarrow F^* \otimes K_2^M(F).$$

Then

- (i) $\Theta(k_{a,b,c} + k_{b,a,c}) = a \otimes \{b, c\} + b \otimes \{a, c\},$
- (ii) $\Theta(\mathbf{c}(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1}))) = 2a \otimes \{b, c\},$
- (iii) $\Theta(k_{c,a,b}) = b \otimes \{a, c\} - a \otimes \{b, c\}.$

Proof. (i) It is easy to see that the exact sequence

$$0 \longrightarrow H_3(SL_3)_{F^*} \longrightarrow H_3(GL_3)/H_3(GL_1) \xrightarrow{\psi} H_1(F^*, H_2(SL_3)) \longrightarrow 0$$

splits canonically by the map

$$F^* \otimes K_2^M(F) = H_1(F^*, H_2(SL_3)) \longrightarrow H_3(GL_3)/H_3(GL_1)$$

defined by Ψ . Now consider the commutative diagram

$$\begin{array}{ccc} H_3(GL_2)/H_3(GL_1) & \xrightarrow{\varphi} & H_1(F^*, H_2(SL_2)) \\ \downarrow & & \downarrow \\ H_3(GL_3)/H_3(GL_1) & \xrightarrow{\psi} & H_1(F^*, H_2(SL_3)). \end{array}$$

We have

$$\begin{aligned} \text{inc}_{1*}(k_{a,b,c}) &= \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, 1)) \\ &= \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})) \\ &\quad + \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, 1, c)) \\ &= \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})) \\ &\quad - \mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, 1, c)) \\ &= \mathbf{c}(\text{diag}(a, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})) \\ &\quad - \mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c^{-1}, c)) \\ &\quad - \mathbf{c}(\text{diag}(b, 1, 1), \text{diag}(1, a, 1), \text{diag}(1, c, 1)) \\ &= \Psi(a \otimes \{b, c\} + b \otimes \{a, c\}) - \text{inc}_{1*}(k_{b,a,c}). \end{aligned}$$

Hence $\text{inc}_{1*}(k_{a,b,c} + k_{b,a,c}) = \Psi(a \otimes \{b, c\} + b \otimes \{a, c\})$. Therefore

$$\begin{aligned} \Theta(k_{a,b,c} + k_{b,a,c}) &= \psi \circ \text{inc}_{1*}(k_{a,b,c} + k_{b,a,c}) \\ &= \psi \circ \Psi(a \otimes \{b, c\} + b \otimes \{a, c\}) \\ &= a \otimes \{b, c\} + b \otimes \{a, c\}. \end{aligned}$$

(ii) Consider the composition

$$F^* \otimes K_2^M(F) \xrightarrow{\Phi} H_3(GL_2)/H_3(GL_1) \xrightarrow{\Theta} F^* \otimes K_2^M(F).$$

The image of $\Phi(a \otimes \{b, c\}) = \mathbf{c}(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1}))$ in the group $H_3(\text{GL}_3)/H_3(\text{GL}_1) = H_3(\text{SL}_3)_{F^*} \oplus F^* \otimes K_2^M(F)$ is equal to

$$\begin{aligned} \text{inc}_{1*} \circ \Phi(a \otimes \{b, c\}) &= \mathbf{c}(\text{diag}(a, a, 1), \text{diag}(b, 1, 1), \text{diag}(c, c^{-1}, 1)) \\ &= \mathbf{c}(\text{diag}(a, a, a^{-2}), \text{diag}(b, 1, 1), \text{diag}(c, c^{-1}, 1)) \\ &\quad + \mathbf{c}(\text{diag}(1, 1, a^2), \text{diag}(b, 1, 1), \text{diag}(c, c^{-1}, 1)) \\ &= \mathbf{c}(\text{diag}(a, 1, a^{-1}), \text{diag}(b, 1, b^{-1}), \text{diag}(c, c^{-1}, 1)) \\ &\quad + \mathbf{c}(\text{diag}(a^2, 1, 1), \text{diag}(1, b, 1), \text{diag}(1, c, c^{-1})). \end{aligned}$$

Therefore $\Theta \circ \Phi(a \otimes \{b, c\}) = \psi \circ \text{inc}_{1*} \circ \Phi(a \otimes \{b, c\}) = 2a \otimes \{b, c\}$.

(iii) First note that

$$\begin{aligned} \Phi(a \otimes \{b, c\}) &= \mathbf{c}(\text{diag}(a, a), \text{diag}(b, 1), \text{diag}(c, c^{-1})) \\ &= \mathbf{c}(\text{diag}(a, 1), \text{diag}(b, 1), \text{diag}(c, 1)) \\ &\quad - k_{c,a,b} + k_{a,b,c} + k_{b,a,c}. \end{aligned}$$

Therefore

$$\begin{aligned} \Theta(k_{c,a,b}) &= \Theta(k_{a,b,c} + k_{b,a,c}) - \Theta(\Phi(a \otimes \{b, c\})) \\ &= b \otimes \{a, c\} - a \otimes \{b, c\}. \end{aligned}$$

□

Proposition 2.3. *Let $\text{inc}_{2*} : H_3(\text{SL}_2)_{F^*} \longrightarrow H_3(\text{GL}_2)$ be induced by the natural map $\text{inc}_2 : \text{SL}_2 \longrightarrow \text{GL}_2$. Then*

$$\text{im}(\text{inc}_{2*}) \cap (F^* \cup H_2(\text{GL}_1)) = \ker(\text{inc}_{1*}).$$

Proof. By Theorem 2.1, the kernel of $\text{inc}_{1*} : H_3(\text{GL}_2) \longrightarrow H_3(\text{GL}_3)$ consists of elements of the form $\sum k_{a,b,c} + k_{b,a,c}$ such that

$$\sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0 \in F^* \otimes K_2^M(F).$$

By Lemma 2.2, we see that

$$\varphi(\sum k_{a,b,c} + k_{b,a,c}) = \sum a \otimes \{b, c\} + b \otimes \{a, c\} = 0.$$

Since the sequence

$$H_3(\text{SL}_2)_{F^*} \xrightarrow{\text{inc}_{2*}} H_3(\text{GL}_2)/H_3(\text{GL}_1) \longrightarrow H_1(F^*, H_2(\text{SL}_2)) \longrightarrow 0,$$

is exact, $\sum k_{a,b,c} + k_{b,a,c} \in \text{im}(\text{inc}_{2*})$. Therefore

$$\ker(\text{inc}_{1*}) \subseteq \text{im}(\text{inc}_{2*}) \cap (F^* \cup H_2(\text{GL}_1)).$$

Now let $x \in \text{im}(\text{inc}_{2*}) \cap (F^* \cup H_2(\text{GL}_1))$. Then x is of the following form

$$x = \sum \mathbf{c}(\text{diag}(a_i, 1), \text{diag}(1, b_i), \text{diag}(1, c_i)).$$

Thus $\det_*(x) = \sum \mathbf{c}(a_i, b_i, c_i) = 0$, where $\det_* : H_3(\mathrm{GL}_2) \rightarrow H_3(F^*)$ is induced by the determinant. By the inclusion $\bigwedge_{\mathbb{Z}}^3 F^* \hookrightarrow H_3(F^*)$, we have $a \wedge b \wedge c \mapsto \mathbf{c}(a, b, c)$ (see for example [10, Lemma 5.5]). Thus

$$\sum a_i \otimes b_i \otimes c_i = \sum a' \otimes a' \otimes b' + \sum a'' \otimes b'' \otimes a'' + \sum b''' \otimes a''' \otimes a'''.$$

Under the composition $F^{*\otimes 3} \rightarrow F^* \otimes H_2(F^*) \rightarrow H_3(\mathrm{GL}_2)$ defined by

$$a \otimes b \otimes c \mapsto a \otimes \mathbf{c}(b, c) \mapsto \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, b), \mathrm{diag}(1, c)) = k_{a,b,c},$$

we see that x has the following form

$$x = \sum \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, b)).$$

For simplicity, we assume that $x = \mathbf{c}(\mathrm{diag}(a, 1), \mathrm{diag}(1, a), \mathrm{diag}(1, b))$. By Lemma 2.2, $\Theta(x) = a \otimes \{a, b\} - b \otimes \{a, a\} = 0$. Thus

$$\begin{aligned} & \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, b, b^{-1})) \\ &= \Psi(a \otimes \{a, b\}) = \Psi(b \otimes \{a, a\}) = \\ & \mathbf{c}(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, a, a^{-1})), \end{aligned}$$

and so

$$\begin{aligned} & + \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, b, 1)) \\ & - \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, b)) \\ &= \\ & - \mathbf{c}(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, a)). \end{aligned}$$

Hence in $H_3(\mathrm{GL}_3)$ we have

$$\begin{aligned} \mathrm{inc}_{1*}(x) &= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, b, 1)) \\ &= \mathbf{c}(\mathrm{diag}(a, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, b)) \\ &\quad - \mathbf{c}(\mathrm{diag}(b, 1, 1), \mathrm{diag}(1, a, 1), \mathrm{diag}(1, 1, a)) \\ &= 0 \end{aligned}$$

Therefore $x \in \ker(\mathrm{inc}_{1*})$ and this completes the proof of the proposition. \square

3. THE INDECOMPOSABLE PART OF $K_3(F)$

Define the *pre-Bloch group* $\mathfrak{p}(F)$ of F as the quotient of the free abelian group $Q(F)$ generated by symbols $[a]$, $a \in F^* - \{1\}$, by the subgroup generated by elements of the form

$$[a] - [b] + \left[\frac{b}{a} \right] - \left[\frac{1-a^{-1}}{1-b^{-1}} \right] + \left[\frac{1-a}{1-b} \right],$$

where $a, b \in F^* - \{1\}$, $a \neq b$. Define

$$\lambda' : Q(F) \rightarrow F^* \otimes F^*, \quad [a] \mapsto a \otimes (1-a).$$

By a direct computation, we have

$$\lambda'([a] - [b] + \left[\frac{b}{a} \right] - \left[\frac{1-a^{-1}}{1-b^{-1}} \right] + \left[\frac{1-a}{1-b} \right]) = a \otimes \left(\frac{1-a}{1-b} \right) + \left(\frac{1-a}{1-b} \right) \otimes a.$$

Let $(F^* \otimes F^*)_\sigma := F^* \otimes F^* / \langle a \otimes b + b \otimes a : a, b \in F^* \rangle$. We denote the elements of $\mathfrak{p}(F)$ and $(F^* \otimes F^*)_\sigma$ represented by $[a]$ and $a \otimes b$ again by $[a]$ and $a \otimes b$, respectively. Thus we have a well-defined map

$$\lambda : \mathfrak{p}(F) \longrightarrow (F^* \otimes F^*)_\sigma, \quad [a] \mapsto a \otimes (1 - a).$$

The kernel of λ is called the *Bloch group* of F and is denoted by $B(F)$. Therefore we obtain the exact sequence

$$0 \longrightarrow B(F) \longrightarrow \mathfrak{p}(F) \longrightarrow (F^* \otimes F^*)_\sigma \longrightarrow K_2^M(F) \longrightarrow 0.$$

The following remarkable theorem is due to Suslin [10, Theorem 5.2].

Theorem 3.1. *Let F be an infinite field. Then we have the exact sequence*

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim} \longrightarrow K_3(F)^{\mathrm{ind}} \longrightarrow B(F) \longrightarrow 0,$$

where $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim}$ is the unique nontrivial extension of the group $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$ by $\mathbb{Z}/2$ if $\mathrm{char}(F) \neq 2$ and is equal to $\mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))$ if $\mathrm{char}(F) = 2$.

The following theorem has been proved in [8, Theorem 4.4].

Theorem 3.2. *Let F be an infinite field. Then we have the exact sequence*

$$0 \longrightarrow \mathrm{Tor}_1^{\mathbb{Z}}(\mu(F), \mu(F))^{\sim} \longrightarrow \tilde{H}_3(\mathrm{SL}_2(F)) \longrightarrow B(F) \longrightarrow 0,$$

where $\tilde{H}_3(\mathrm{SL}_2(F)) := H_3(\mathrm{GL}_2) / (H_3(\mathrm{GL}_1) + F^* \cup H_2(\mathrm{GL}_1))$.

These two theorems suggest that $K_3(F)^{\mathrm{ind}}$ and $\tilde{H}_3(\mathrm{SL}_2(F))$ should be isomorphism. But there is no natural homomorphism from one of these groups to the other one! But there is a natural maps from $H_3(\mathrm{SL}_2)_{F^*}$ to both of them. Hutchinson and Tao have proved that $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow K_3(F)^{\mathrm{ind}}$ is surjective [4, Lemma 5.1]. The next lemma claims that this is also true for the other map.

Lemma 3.3. *The map $\varsigma : H_3(\mathrm{SL}_2)_{F^*} \longrightarrow \tilde{H}_3(\mathrm{SL}_2)$, induced by the natural map $\mathrm{SL}_2 \longrightarrow \mathrm{GL}_2$, is surjective.*

Proof. Consider the exact sequence

$$H_3(\mathrm{SL}_2)_{F^*} \longrightarrow H_3(\mathrm{GL}_2) / H_3(\mathrm{GL}_1) \xrightarrow{\varphi} H_1(F^*, H_2(\mathrm{SL}_2)) \longrightarrow 0.$$

By Lemma 2.2, we have

$$\begin{aligned} \Theta(k_{a,b,c} + k_{b,a,c}) &= a \otimes \{b, c\} + b \otimes \{a, c\}, \\ \Theta(\mathbf{c}(\mathrm{diag}(a, a), \mathrm{diag}(b, 1), \mathrm{diag}(c, c^{-1}))) &= 2a \otimes \{b, c\}. \end{aligned}$$

Since $H_1(F^*, H_2(\mathrm{SL}_2))$ as a subgroup of $H_1(F^*, H_2(\mathrm{SL}_3)) = F^* \otimes K_2^M(F)$ is generated by elements $a \otimes \{b, c\} + b \otimes \{a, c\}$ and $2d \otimes \{e, f\}$ and since $a \cup \mathbf{c}(b, c) = k_{a,b,c}$ vanish in $\tilde{H}_3(\mathrm{SL}_2)$, $H_3(\mathrm{SL}_2)_{F^*} \longrightarrow \tilde{H}_3(\mathrm{SL}_2)$ must be surjective. \square

Now we are ready to prove our main theorem.

Theorem 3.4. *Let F be an infinite field. The following conditions are equivalent.*

- (i) *The homomorphism $\alpha : H_3(SL_2)_{F^*} \rightarrow K_3(F)^{\text{ind}}$ is bijective.*
- (ii) *The natural homomorphisms $\text{inc}_{1*} : H_3(GL_2) \rightarrow H_3(GL_3)$ and $\text{inc}_{2*} : H_3(SL_2)_{F^*} \rightarrow H_3(GL_2)$ are injective.*

Proof. (ii) \Rightarrow (i) Consider the surjective map $\varsigma : H_3(SL_2)_{F^*} \rightarrow \tilde{H}_3(SL_2)$ from Lemma 3.3. Let $\varsigma(x) = 0$. Then $\text{inc}_{2*}(x) \in \text{im}(\text{inc}_{2*}) \cap F^* \cup H_3(GL_1)$. But by Proposition 2.3 and the assumptions

$$\text{im}(\text{inc}_{2*}) \cap F^* \cup H_3(GL_1) = \ker(\text{inc}_{1*}) = 0.$$

From this we have $\text{inc}_{2*}(x) = 0$ and hence $x = 0$. Therefore ς is an isomorphism. Now the claim follows by comparing the exact sequence of Theorem 3.2 and Suslin's Bloch-Wigner exact sequence in Theorem 3.1.

(i) \Rightarrow (ii) Let \bar{F} be the algebraic closure of F . By a theorem of Merkurjev and Suslin, $K_3(F)^{\text{ind}} \rightarrow K_3(\bar{F})^{\text{ind}}$ is injective [5, Proposition 11.3]. Thus from the commutative diagram

$$\begin{array}{ccc} H_3(SL_2)_{F^*} & \longrightarrow & H_3(SL_2(\bar{F})) \\ \downarrow & & \downarrow \simeq \\ K_3(F)^{\text{ind}} & \longrightarrow & K_3(\bar{F})^{\text{ind}}, \end{array}$$

and the injectivity of α , we deduce the injectivity of the map $H_3(SL_2)_{F^*} \rightarrow H_3(SL_2(\bar{F}))$. Now the injectivity of $H_3(SL_2)_{F^*} \rightarrow H_3(GL_2)$ follows from the injectivity of $H_3(SL_2(\bar{F})) \rightarrow H_3(GL_2(\bar{F}))$ [6, Theorem 6.1] and commutativity of the diagram

$$\begin{array}{ccc} H_3(SL_2)_{F^*} & \longrightarrow & H_3(SL_2(\bar{F})) \\ \downarrow & & \downarrow \\ H_3(GL_2) & \longrightarrow & H_3(GL_2(\bar{F})). \end{array}$$

On the other hand, by Proposition 2.3, $\ker(\text{inc}_{1*}) \subseteq H_3(SL_2)_{F^*} \subseteq H_3(GL_2)$. Let $\text{inc}_{1*}(x) = 0$. It easily follows from the commutative diagram

$$\begin{array}{ccccc} H_3(SL_2)_{F^*} & \longrightarrow & H_3(GL_2) & \longrightarrow & H_3(GL_3) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(SL_2(\bar{F})) & \longrightarrow & H_3(GL_2(\bar{F})) & \longrightarrow & H_3(GL_3(\bar{F})). \end{array}$$

that $x \in \ker(H_3(GL_2(\bar{F})) \rightarrow H_3(GL_3(\bar{F}))) = 0$. Therefore $x = 0$ [6, Theorem 5.4(iii)]. \square

Remark 3.5. (i) From the spectral sequence $\mathcal{E}_{p,q}^2$, one get the exact sequence

$$\begin{aligned} H_4(GL_2)/H_4(GL_1) &\rightarrow H_2(F^*, H_2(SL_2)) \xrightarrow{\vartheta_{2,2}^2} H_3(SL_2)_{F^*} \\ &\rightarrow H_3(GL_2)/H_3(GL_1) \rightarrow H_1(F^*, H_2(SL_2)) \rightarrow 0. \end{aligned}$$

Thus the injectivity of inc_{2*} is equivalent to triviality of the differential $\mathfrak{d}_{2,2}^2$.

(ii) Theorem 2.1 gives a clear description of elements of the kernel of inc_{1*} . But there is no such information about the kernel of inc_{2*} . It is easy to see that $s_{a,b,c} := \mathbf{c}(\text{diag}(a, a^{-1}), \text{diag}(b, b^{-1}), \text{diag}(c, c^{-1}))$ is in the kernel of inc_{2*} and is 2-torsion:

$$\begin{aligned} s_{a,b,c} &= \mathbf{c}(\text{diag}(a, a^{-1}), \text{diag}(b, b^{-1}), \text{diag}(c, c^{-1})) \\ &= \mathbf{c}(w.\text{diag}(a, a^{-1}).w^{-1}, w.\text{diag}(b, b^{-1}).w^{-1}, w.\text{diag}(c, c^{-1}).w^{-1}) \\ &= \mathbf{c}(\text{diag}(a^{-1}, a), \text{diag}(b^{-1}, b), \text{diag}(c^{-1}, c)) \\ &= -s_{a,b,c}, \end{aligned}$$

where $w := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. But it is not clear to us why it should be zero.

It is not difficult to see that $\ker(\text{inc}_{2*})$ is a 2-power torsion group (see for example the proof of Theorem 6.1 in [6]).

REFERENCES

- [1] Bloch, S. J. Higher Regulators, Algebraic K -Theory and Zeta Functions of Elliptic Curves. CRM Monograph Series. 11. Providence, RI: American Mathematical Society (AMS), 2000. [1](#)
- [2] Brown, K. S. Cohomology of Groups. Graduate Texts in Mathematics, 87. Springer-Verlag, New York, 1994. [4](#)
- [3] Elbaz-Vincent, P. The indecomposable K_3 of rings and homology of SL_2 . J. Pure Appl. Algebra **132** (1998), no. 1, 27–71. [4](#)
- [4] Hutchinson, K., Tao, L. The third homology of the special linear group of a field. J. Pure Appl. Algebra **213** (2009), no. 9, 1665–1680. [1](#), [2](#), [3](#), [8](#)
- [5] Merkurjev, A. S., Suslin, A. A. The group K_3 for a field. Math. USSR-Izv. **36** (1991), no. 3, 541–565. [9](#)
- [6] Mirzaii, B. Third homology of general linear groups. J. Algebra. **320** (2008), no. 5, 1851–1877. [2](#), [9](#), [10](#)
- [7] Mirzaii, B. Third homology of general linear groups over rings with many units. Preprint, available at <http://arxiv.org/abs/1108.5454> [2](#), [4](#)
- [8] Mirzaii, B., Yeganeh, Fatemeh. Bloch-Wigner theorem over rings with many units II. Preprint, available at <http://arxiv.org/abs/1108.5452> [8](#)
- [9] Sah, C. Homology of classical Lie groups made discrete. III. J. Pure Appl. Algebra **56** (1989), no. 3, 269–312. [1](#)
- [10] Suslin, A. A. K_3 of a field and the Bloch group. Proc. Steklov Inst. Math. 1991, **183** no. 4, 217–239. [1](#), [7](#), [8](#)

Department of Mathematics,
Institute for Advanced Studies in Basic Sciences,
P. O. Box. 45195-1159, Zanjan, Iran.
email: bmirzaii@iasbs.ac.ir